What is a wave? It is a physical process whereby energy (or information about boundary conditions or forcing) is transported through a medium without any significant transport of the material in that medium. Each kind of wave is characterized by a **restoring force** that leads to the oscillatory behavior. For example, for waves on the surface of the ocean, the restoring force is gravity…if you raise a water parcel from the sea surface and release it, gravity tends to pull it down again.

A simple wave form is the propagating (or progressive) sinusoidal wave in only one spatial dimension, described by the function for the displacement of a surface (say the sea surface) as a function of space and time:

$$\eta(x,t) = a\cos(kx - \omega t)$$

See Knauss, Figure 9.2 for a diagram of this function. Here are some of the basic quantities that describe a general wave.

1. Amplitude, a = _ the distance from crest to trough. **Wave height,** $\boldsymbol{H}$ = distance from crest to trough, $H = 2a$.

2. Wavelength of wave, Λ. Related is the **wavenumber** $k = 2\pi/\Lambda$._

3. Period of wave, T. Related to **frequency** $\omega = 2\pi/T$. Units are radians per second. Alternate units are cycles/sec. 2π radians = 1 cycle.

4. Phase, $\varphi = kx\text{-}\omega t$. Phase is a measure of the point within a wave cycle. At a given x and t, the phase will tell you whether or not you are at a wave crest, a trough or someplace in between. Phase varies from 0 to 2π and then repeats.

Crests occur when $\varphi = 0, 2\pi, 4\pi, \ldots$ (where $\cos\varphi = 1$).

Troughs occur when $\varphi = \pi, 3\pi, 5\pi, \ldots$ (where $\cos\varphi = -1$).

Surface displacement goes through zero at $\varphi = \pi/2, 3\pi/2, 5\pi/2, \ldots$

5. Phase velocity (phase speed if one spatial dimension) quantifies how fast crests (or troughs) travel. Following a crest, the phase is constant (same point in wave cycle) so $kx\text{-}\omega t$ = const. This will occur if the speed of travel $\Delta x/\Delta t = \omega/k$. Thus we define phase speed (C for "celerity") as

$$C = \omega/k = \Lambda/T.$$

6. Dispersion relation. For the various kinds of theoretical waves there is an analytic relationship between ω and k. This is called the **dispersion relationship** $\omega = f(k)$.
Based on the dispersion relation waves are classified into two kinds:

(i) Nondispersive: the dispersion relationship $\omega = \text{const}*k$. All wave numbers and frequencies have the same phase speed

(ii) Dispersive: $\omega = F(k)$. $C = \omega/k = F(k)/k$. Different wave numbers have different phase speeds.

7. Standing waves

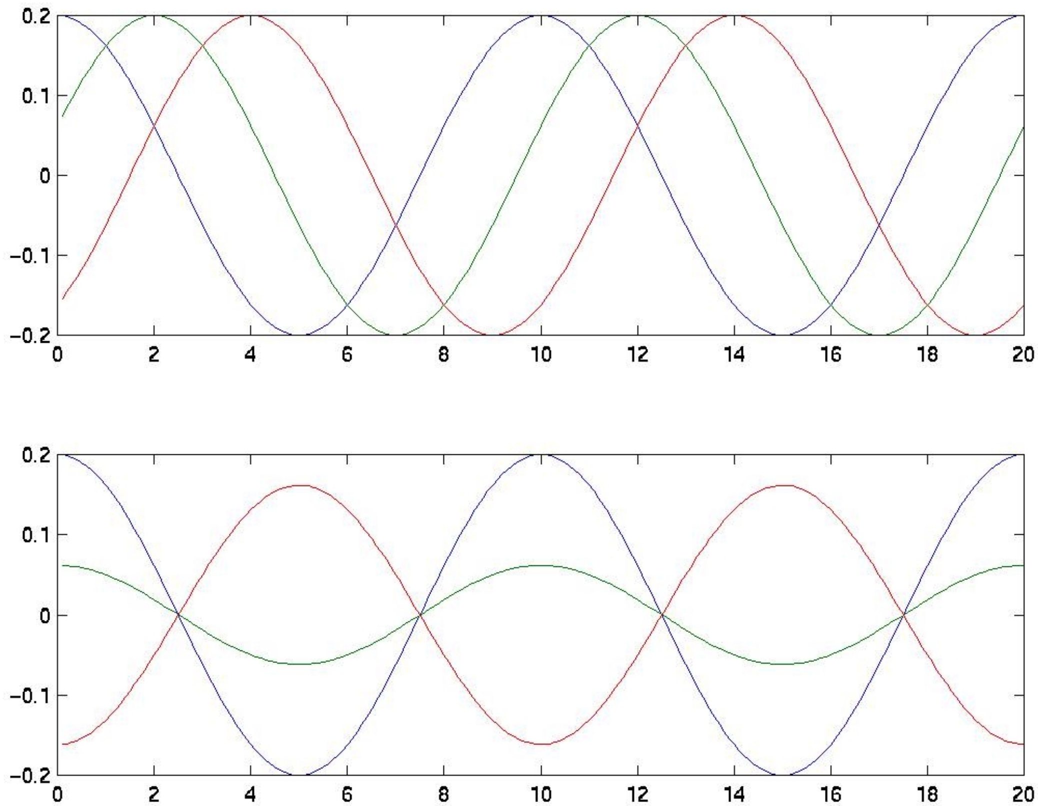
Waves can also have a different form, particularly in a finite domain, and that is a standing wave. A standing wave composed of two waves traveling in opposite directions with equal amplitudes the same wave number and frequency. That is

$$\eta(x,t) = \frac{a}{2}\cos(kx - \omega t) + \frac{a}{2}\cos(kx + \omega t)$$

Using trigonometric identies, we can write this as

$$\eta(x,t) = a\cos(kx)\cos(\omega t)$$

In case, the wave does not propagate, it gets larger or smaller in place. Shown in the next figure waves forms. The top picture is for a propagating wave at three different times (blue at t=0, green at t=2 and red at t=4 for a period 10 wave) as a function of x. The lower pictures is for a standing wave at the same times.



8. **Group velocity,** $C_g = \frac{\partial \omega}{\partial k}$ = velocity at which the wave propagates energy. Why is this different from phase velocity? First, how do we define energy? It can be different for different waves, but generally it is proportional to the square of the sea surface displacement. Energy ~ a^2. Any real forcing will generate waves at several, often fairly closely related frequencies, and since there is a relationship between frequency and wavenumber, the forced waves will also have several different wavelengths. These different wave components can, if the waves are dispersive, travel at different speeds. These dispersing waves interfere constructively and destructively to produce a pattern of sea-surface displacement that can be quite complicated. The group velocity

is the rate at which the surface displacement function made up from the sum of all the different wave components travels.

Let's look at the simplest possible case - two waves traveling in the same direction at slightly different wavenumbers and frequencies. At any given time, there will be points in space where the waves constructively interfere to produce an oscillation twice as strong -- and there will be points in space where the waves destructively interfere to produce a very low amplitude. See Knauss, Figures 9.7 and 9.8. The packets of constructive interference are termed wave "groups". Both waves are moving off to the right in Fig. 9.8 and the wave groups will also propagate to the right -- but at a much slower rate than the individual wave crests.

For Figure 9.8, the group velocity is the rate at which the sum:

$$\eta = a\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$

translates to the right. This occurs at a speed = $\frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\Delta\omega}{\Delta k}$. See Knauss, Box 9.2 for the derivation. The sum of the two waves is an oscillation modulated by a wave of wavenumber Δk and frequency $\Delta\omega$.

Real forcing often generates waves over a small range of frequencies and wavenumbers, so that Δk and $\Delta\omega$ are small. The result is that the modulation has a much longer wavelength and longer period than the individual waves that make it up. Since we often have a theoretical functional relationship, the dispersion relationship, between ω and k, the group velocity is defined in a continuous sense as $C_g = \frac{\partial\omega}{\partial k}$.

Notation alert:

	Knauss	Me
Phase speed = ω/k	C	C
Group speed = $\partial\omega/\partial k$	Curly "V"	C_g

Note: a difference between phase and group velocities only occurs when the phase speed is not a constant.

Nondispersive waves: $\omega = C_0 k$, where C_0 is a constant. Then, $\omega/k = \partial\omega/\partial k = C_0$ = constant. Group velocity = phase velocity.

Dispersive waves: $\omega = F(k)$. $C = \omega/k = F(k)/k$. $C_g = \frac{\partial F}{\partial k}$. Group and phase velocities are different.

For the remainder of the course, we will be considering the various kinds of waves that are found in the ocean. These various waves are distinguished by:

(1) Their timescale. See Knauss, Figure 9.1. The **spectrum** is a plot of the distribution of energy as a function of frequency. How would we measure a wave spectrum? Have a timeseries at one point, say from a pressure gauge on the bottom of the ocean. Through a process called Fourier decomposition, we can separate out the sines and cosines at different frequencies that add up together to make a general signal. The amplitude (squared) of an individual wave component is proportional to the energy at that frequency.

(2) Their basic dynamics. (What are the important terms in $F = ma$?)

(3) The dispersion relationship $\omega = F(k)$. How the wave moves energy and information around in the ocean.

(4) The form of the solution. What the wave looks like, in terms of variables we might be interested in measuring such as η (surface displacement), p (pressure), $\underline{u}$ (particle velocity), etc.

Generally we are not going to deal with the details of the manipulation of the equations to get the solutions. We will focus on the basic balances and then we will work with describing the solutions and dispersion relationship, and the role of these waves in ocean processes.

Questions:

1. If a wave has period 10 s, what is its frequency ω?
2. If a wave has wavelength of 100m, what is its wavenumber k?
3. What is the phase speed for the wave described in 1 and 2 above?
4. In equation 9.16, Knauss writes that the phase speed for capillary waves is given by

$$C = \sqrt{\frac{g}{k} + \frac{k}{\rho_0}\varsigma}$$

Thus for these wave $\omega = \sqrt{gk + k^3 \frac{\varsigma}{\rho_0}}$

Here ς is the surface tension and is $0.074\, kg/s^2$. You can read about capillary waves on pages 210 and 211 of the textbook.

   a) Is the wave dispersive or non-dispersive?
   b) Derive a formula for the group velocity as a function of wave number for capillary waves.
   c) Now calculate the phase and group velocities for waves with wavelength 25 cm and 1 mm.
   d) Is the group velocity smaller or bigger than the phase velocity for the 25cm wave? How about the 1mm wave?